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A method for construction of the matrix solvable models

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Abstract

We give a method for construction of the solvable models. The method is based on an idea of Turbiner's to transform the Hamiltonian into a simple solvable form. This method gives a possibility of constructing some matrix models. We apply this method to the models of the Calogero type and obtain solvable models of the three-particle system with the matrix 2×2 potential, which was constructed by Polychronakos, and a new model with the matrix 3×3 potential.

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1. Introduction

We deal with an algebraic method for the eigenvalue problem for the differential operator

$$\mathbf{H} = \eta^{ik}\partial_{ik} - \mathbf{U}(x), \tag{1}$$

where η^{ik} is a constant symmetric matrix, $\mathbf{U}(x)$ is a symmetric matrix function of the variables x^i, $i = 1, \ldots, n$, $\partial_i = \frac{\partial}{\partial x^i}$.

We use an idea from some of Turbiner's papers (see [1] and references therein) in which the authors try to rewrite the differential operator

$$\widehat{\mathbf{H}} = g^{rs}(y)\partial_{rs} + b^r(y)\partial_s + V(y) \tag{2}$$

by the realization of the Lie algebra by differential operators of the first-order derivatives. If the representation on space of the polynomials has invariant finite-dimensional subspaces, it is possible to find the spectrum of this operator, or part of its spectrum, respectively, by pure algebraic calculations. In the first case Turbiner calls the models exact solvable and in the second case quasi-solvable.

To transform equation (1) to (2) the transformations $y^r = y^r(x^k)$ and $\psi = e^{a(x)}\widehat{\psi}$ are used. In this case, there are relations between functions $g^{rs}(y)$, $b^r(y)$, $V(y)$ and $y^r(x)$, $a(x)$, $U(x)$.

If the functions g^{rs}, b^r and V fulfil the same conditions, it is possible from these relations to reconstruct the operator of type (1) from (2).

In the paper, we generalize this method to the case of a matrix potential. Some notes about this generalization can be found, for instance, in paper [2]. Moreover, we do not use the realizations of the Lie algebra. In general, we suppose the form of the differential operator (2) for which the invariant finite-dimensional space of polynomials is evident. We also apply the derived general relations to the case of systems connected with the three-particle Calogero systems [3].

2. The general construction

Let us consider the differential operator (1), where $\mathbf{U}(x)$ is a real symmetric matrix function of type $N \times N$ and $\psi(x)$ is an N-component vector wavefunction. First, we introduce the new 'wavefunction' $\widehat{\psi}$ by the relation $\psi(x) = \mathbf{G}(x)\widehat{\psi}(x)$, where $\mathbf{G}(x)$ is a regular matrix function of the variables $x^k, k = 1, 2, \ldots, n$. The differential operator $\mathbf{H}$ transforms by the relation $\mathbf{G}^{-1}\mathbf{H}\mathbf{G}\widehat{\psi} = \widehat{\mathbf{H}}\widehat{\psi}$, where

$$\widehat{\mathbf{H}} = \eta^{ik}\partial_{ik} + 2\eta^{ik}\mathbf{X}_i\partial_k + \eta^{ik}\mathbf{X}_{i,k} + \eta^{ik}\mathbf{X}_k\mathbf{X}_i - \widehat{\mathbf{U}} \tag{3}$$

and

$$\partial_i\mathbf{G} = \mathbf{G}\mathbf{X}_i, \tag{4}$$

with $\mathbf{X}_{i,k} = \partial_k\mathbf{X}_i$ and $\widehat{\mathbf{U}} = \mathbf{G}^{-1}\mathbf{U}\mathbf{G}$.

By derivation of equation (4) we obtain for the functions $\mathbf{X}_i$ the compatibility condition

$$\partial_k(\mathbf{X}_i) - \partial_i(\mathbf{X}_k) = \mathbf{X}_i\mathbf{X}_k - \mathbf{X}_k\mathbf{X}_i = [\mathbf{X}_i, \mathbf{X}_k]. \tag{5}$$

Now we introduce new coordinates $y^r = y^r(x^i), r = 1, 2, \ldots, n$, where the functions $y^r(x^i)$ define smooth regular mapping and denote

$$q_i^r = \frac{\partial y^r}{\partial x^i} \qquad \text{and} \qquad p_r^i = \frac{\partial x^i}{\partial y^r}.$$

With the use of the relations $q_i^r p_s^i = q_s^i p_i^r = \delta_s^r$ we can rewrite (3) in the coordinates y^r as

$$\widehat{\mathbf{H}} = g^{rs}\partial_{rs} + \left(2g^{rs}\mathbf{Y}_s - g^{st}\Gamma_{st}^r\right)\partial_r + g^{rs}\left(\mathbf{Y}_{r,s} + \mathbf{Y}_s\mathbf{Y}_r - \Gamma_{rs}^t\mathbf{Y}_t\right) - \widehat{\mathbf{U}}, \tag{6}$$

where ∂_r denotes derivation according to the variable y^r, $g^{rs} = \eta^{ik}q_i^r q_k^s$ is the metric tensor, $\Gamma_{st}^r = -p_s^i p_t^k q_{ik}^r$ is the corresponding connection, $\mathbf{Y}_s = p_s^i\mathbf{X}_i$ and $\mathbf{Y}_{r,s} = \partial_s\mathbf{Y}_r$. Thus, after these transformations we obtain the operator $\widehat{\mathbf{H}}$ in the form

$$\widehat{\mathbf{H}} = g^{rs}(y)\partial_{rs} + 2\mathbf{b}^r(y)\partial_r + \mathbf{V}(y), \tag{7}$$

where

$$\mathbf{b}^r = g^{rs}\mathbf{Y}_s - \tfrac{1}{2}g^{st}\Gamma_{st}^r, \tag{8}$$

$$\mathbf{V} = g^{rs}\left(\mathbf{Y}_{r,s} + \mathbf{Y}_s\mathbf{Y}_r - \Gamma_{rs}^t\mathbf{Y}_t\right) - \widehat{\mathbf{U}}. \tag{9}$$

If the functions $g^{rs}(y)$, $\mathbf{b}^r(y)$ and $\mathbf{V}(y)$ have a special form, invariant finite-dimensional subspaces of functions for the differential operator (7) are evident. In this case, it is possible to find any eigenvalues and eigenfunctions of the operator $\widehat{\mathbf{H}}$ and, consequently, the operator $\mathbf{H}$. For example, if the functions $g^{rs}(y)$ are quadratic polynomials in variables y^r, $\mathbf{b}^r(y)$ are matrix linear functions, and $\mathbf{V}$ is a constant matrix, the set of polynomials of order less than or equal to M is for any M invariant subspace. But there are other cases when the invariant subspace of the operator (7) is evident.

Now, we deal with necessary conditions which have to fulfil the functions g^{rs}, $\mathbf{b}^r$ and $\mathbf{V}$ to be possible to obtain (7) from (1) by the above-mentioned process.

First, a transformation of the variables $x^k = x^k(y^r)$ has to exist, for which

$$g^{rs} p^i_r p^k_s = \eta^{ik}. \tag{10}$$

It is very well known that the conditions for integrability of this system of differential equations are

$$R^k_{rst} = \partial_t \Gamma^k_{rs} - \partial_s \Gamma^k_{rt} + \Gamma^i_{rs}\Gamma^k_{it} - \Gamma^i_{rt}\Gamma^k_{is} = 0,$$

where

$$\Gamma^r_{st} = g^{rk}\Gamma_{st,k} \qquad \Gamma_{st,k} = \tfrac{1}{2}(-\partial_k g_{st} + \partial_s g_{tk} + \partial_t g_{sk}),$$

and g_{rs} is the inverse matrix to g^{rs}, i.e. $g^{rs} g_{st} = \delta^r_t$.

According to (8),

$$\mathbf{Y}_r = \mathbf{b}_r + \tfrac{1}{2}\Gamma_r, \tag{11}$$

where $\mathbf{b}_r = g_{rs}\mathbf{b}^s$ and $\Gamma_r = g^{st}\Gamma_{st,r}$, hold. Moreover, from the definition of the matrix function $\mathbf{Y}_r$ we obtain

$$\frac{\partial \mathbf{G}}{\partial y^r} = p^k_r \frac{\partial \mathbf{G}}{\partial x^k} = p^k_r \mathbf{G}\mathbf{X}_k = \mathbf{G}\mathbf{Y}_r.$$

Therefore, the matrix functions $\mathbf{Y}_r$ are solutions of the system of differential equations

$$\partial_r \mathbf{G}(y) = \mathbf{G}(y)\mathbf{Y}_r.$$

The integrability conditions of this system follow from the equations $\partial_s(\partial_r \mathbf{G}) = \partial_r(\partial_s \mathbf{G})$, namely, they are

$$\partial_s \mathbf{Y}_r - \partial_r \mathbf{Y}_s = \mathbf{Y}_r\mathbf{Y}_s - \mathbf{Y}_s\mathbf{Y}_r = [\mathbf{Y}_r, \mathbf{Y}_s]. \tag{12}$$

If we write $\mathbf{b}_r = \widehat{\mathbf{b}}_r + T_r$, where $T_r = \frac{1}{N}\operatorname{Tr}\mathbf{b}_r$, system (12) is equivalent to the equations

$$\partial_s\big(T_r + \tfrac{1}{2}\Gamma_r\big) - \partial_r\big(T_s + \tfrac{1}{2}\Gamma_s\big) = 0 \tag{13}$$

$$\partial_s \widehat{\mathbf{b}}_r - \partial_r \widehat{\mathbf{b}}_s = [\widehat{\mathbf{b}}_r, \widehat{\mathbf{b}}_s]. \tag{14}$$

Equations (13) indicate that there is a function $F(y)$ for which

$$T_r(y) + \tfrac{1}{2}\Gamma_r(y) = \partial_r F(y) = F_{,r}. \tag{15}$$

If we define the matrix function $\widehat{\mathbf{G}}$ by the relation

$$\mathbf{G} = \mathrm{e}^F \widehat{\mathbf{G}},$$

we obtain for this matrix function the linear homogeneous system of the differential equations

$$\partial_r \widehat{\mathbf{G}} = \widehat{\mathbf{G}}\,\widehat{\mathbf{b}}_r. \tag{16}$$

Equation (9) gives

$$\widehat{\mathbf{U}} = \partial_r \widehat{\mathbf{b}}^r + g_{rs}\widehat{\mathbf{b}}^r\widehat{\mathbf{b}}^s + \Gamma^s_{rs}\widehat{\mathbf{b}}^r + 2\widehat{\mathbf{b}}^r F_{,r} + g^{rs}(F_{,rs} + F_{,r}F_{,s} - \Gamma_s F_{,r}) - \mathbf{V} \tag{17}$$

and the function $\mathbf{U}$ in (1) is given by the relation

$$\mathbf{U} = \mathbf{G}\widehat{\mathbf{U}}\mathbf{G}^{-1} = \widehat{\mathbf{G}}\widehat{\mathbf{U}}\widehat{\mathbf{G}}^{-1}. \tag{18}$$

This matrix function could be symmetric. This is an additional condition to the functions $\widehat{\mathbf{b}}^r$ and $\mathbf{V}$. The necessary condition for this assumption can be formulated without knowledge of the function $\widehat{\mathbf{G}}$. If we denote $\mathbf{e}_{ik}$ by the matrices with the components $(\mathbf{e}_{ik})_{rs} = \delta_{ir}\delta_{ks}$, it is possible to formulate the condition for the symmetry matrix $\mathbf{U}$ by the equation $\mathrm{Tr}(\mathbf{U}(\mathbf{e}_{ik} - \mathbf{e}_{ki})) = 0$, which holds for any $i, k = 1, 2, \ldots, N$. Therefore, for any i and k the relation

$$\mathrm{Tr}(\mathbf{U}(\mathbf{e}_{ik} - \mathbf{e}_{ki})) = \mathrm{Tr}(\widehat{\mathbf{G}}\widehat{\mathbf{U}}\widehat{\mathbf{G}}^{-1}(\mathbf{e}_{ik} - \mathbf{e}_{ki})) = \mathrm{Tr}(\widehat{\mathbf{U}}\widehat{\mathbf{G}}^{-1}(\mathbf{e}_{ik} - \mathbf{e}_{ki})\widehat{\mathbf{G}}) = 0$$

holds. If we denote $\mathbf{X}_{ik}^T = \widehat{\mathbf{G}}^{-1}(\mathbf{e}_{ik} - \mathbf{e}_{ki})\widehat{\mathbf{G}}$, these equations can be read as

$$\mathrm{Tr}\big(\widehat{\mathbf{U}}\mathbf{X}_{ik}^T\big) = 0 \qquad \text{for any} \quad i, k. \tag{19}$$

Let us denote by $\mathcal{V}$ the set of matrix functions, for which the relation (19) holds. It is evident that if $\mathbf{M} \in \mathcal{V}$ then $f(y)\mathbf{M} \in \mathcal{V}$ for any function $f(y)$. Moreover, if $\mathbf{M} \in \mathcal{V}$, we obtain by derivation of the equation $\mathrm{Tr}(\mathbf{M}\mathbf{X}_{ik}) = 0$ with respect to variable y^r, according to (16), the relation

$$D_r\mathbf{M} = \partial_r\mathbf{M} + [\widehat{\mathbf{b}}_r, \mathbf{M}] \in \mathcal{V}.$$

It is easy to see that for any $\mathbf{M} \in \mathcal{V}$ and any smooth function $f(y)$ is

$$D_r(f(y)\mathbf{M}) = \mathbf{M}\partial_r f(y) + f(y)D_r\mathbf{M} \in \mathcal{V}.$$

This $\mathcal{V}$ is linear space with the coefficient from the space of the smooth functions. By definition and Jacobi identity it is easy to see that

$$D_s(D_r\mathbf{M}) = D_r(D_s\mathbf{M}) = D_{rs}\mathbf{M}.$$

Since the identity matrices $\mathbf{I}$ and $\widehat{\mathbf{U}}$ are elements of $\mathcal{V}$, the matrices $D_r\widehat{\mathbf{U}}$, $D_{rs}\widehat{\mathbf{U}}$, $D_{rst}\widehat{\mathbf{U}}, \ldots$ are elements of space $\mathcal{V}$, too. But in the space $\mathcal{V}$ there can be only $\frac{N(N+1)}{2}$ linearly independent functions. Therefore, for the potential $\mathbf{U}$ to be symmetric, a large number of matrix functions has to be linearly dependent. From this we obtain additional conditions for the matrices $\widehat{\mathbf{b}}^r$ and $\mathbf{V}$.

3. Three-particle models of the Calogero type

In this section we deal with the Hamiltonian operator

$$\mathbf{H} = -\tfrac{1}{2}(\partial_{11} + \partial_{22} + \partial_{33}) + \mathbf{U}(x_1 - x_2, x_3 - x_2), \tag{20}$$

where $x_1 > x_2 > x_3$ and $\mathbf{U}$ is a symmetric matrix function of the type $N \times N$. It is seen from the form of the potential $\mathbf{U}$ that it is convenient to introduce the relative coordinate by the relations

$$\begin{aligned} X &= x_1 + x_2 + x_3 & x_1 &= y_1 + \frac{1}{3}X \\ y_1 &= x_1 - \frac{1}{3}X = \frac{2x_1 - x_2 - x_3}{3} & x_2 &= \frac{1}{3}X - y_1 - y_2 \\ y_2 &= x_3 - \frac{1}{3}X = \frac{2x_3 - x_1 - x_2}{3} & x_3 &= y_2 + \frac{1}{3}X. \end{aligned}$$

After this substitution the operator (20) transforms to $\mathbf{H} = -\frac{3}{2}\partial_{XX} + \mathbf{H}_{\mathrm{rel}}$, where

$$\mathbf{H}_{\mathrm{rel}} = -\tfrac{1}{3}(\partial_{11} - \partial_{12} + \partial_{22}) + \mathbf{U}_{\mathrm{rel}}(y_1, y_2),$$

and $\mathbf{U}_{\mathrm{rel}}(y_1, y_2) = \mathbf{U}(2y_1 + y_2, y_1 + 2y_2)$. To apply the general method given in the previous section, we study the operator

$$\mathbf{h} = -2\mathbf{H}_{\mathrm{rel}} = \tfrac{2}{3}(\partial_{11} - \partial_{12} + \partial_{22}) - 2\mathbf{U}_{\mathrm{rel}}. \tag{21}$$

This is the operator of the type (1), where $\eta^{11} = \eta^{22} = \frac{2}{3}$, $\eta^{12} = -\frac{1}{3}$ and $\mathbf{U} = 2\mathbf{U}_{\mathrm{rel}}$.

If we introduce the new variables by the relations

$$z_1 = -y_1^2 - y_1y_2 - y_2^2 \qquad \text{and} \qquad z_2 = -y_1y_2(y_1 + y_2),$$

we obtain the metric tensor

$$g^{11} = -2z_1, \quad g^{12} = -3z_2, \quad g^{22} = \tfrac{2}{3}z_1^2, \tag{22}$$

and

$$g_{11} = \frac{-2z_1^2}{4z_1^3 + 27z_2^2}, \qquad g_{12} = \frac{-9z_2}{4z_1^3 + 27z_2^2}, \qquad g_{22} = \frac{6z_1}{4z_1^3 + 27z_2^2}. \tag{23}$$

Since the functions g^{rs} are the quadratic polynomials in the variables z_1, z_2, we suppose that $\mathbf{b}^1$ and $\mathbf{b}^2$ are linear polynomials and $\mathbf{V}$ is a constant matrix, i.e., we suppose

$$\mathbf{b}^1 = \mathbf{C}_0^1 + \mathbf{C}_1^1 z_1 + \mathbf{C}_2^1 z_2, \qquad \mathbf{b}^2 = \mathbf{C}_0^2 + \mathbf{C}_1^2 z_1 + \mathbf{C}_2^2 z_2, \tag{24}$$

where $\mathbf{C}_s^r$ are constant matrices of the type $N \times N$. Therefore

$$\mathbf{b}_1 = -\frac{2z_1^2\left(\mathbf{C}_0^1 + z_1\mathbf{C}_1^1 + z_2\mathbf{C}_2^1\right)}{4z_1^3 + 27z_2^2} - \frac{9z_2\left(\mathbf{C}_0^2 + z_1\mathbf{C}_1^2 + z_2\mathbf{C}_2^2\right)}{4z_1^3 + 27z_2^2}$$

$$\mathbf{b}_2 = -\frac{9z_2\left(\mathbf{C}_0^1 + z_1\mathbf{C}_1^1 + z_2\mathbf{C}_2^1\right)}{4z_1^3 + 27z_2^2} + \frac{6z_1\left(\mathbf{C}_0^2 + z_1\mathbf{C}_1^2 + z_2\mathbf{C}_2^2\right)}{4z_1^3 + 27z_2^2}.$$

If we denote $T_s^r = \frac{1}{N}\operatorname{Tr}\mathbf{C}_s^r$ and $\widehat{\mathbf{C}}_s^r = \mathbf{C}_s^r - T_s^r$, i.e. $\operatorname{Tr}\widehat{\mathbf{C}}_s^r = 0$, we have

$$T_1 = -\frac{2z_1^2\left(T_0^1 + z_1T_1^1 + z_2T_2^1\right)}{4z_1^3 + 27z_2^2} - \frac{9z_2\left(T_0^2 + z_1T_1^2 + z_2T_2^2\right)}{4z_1^3 + 27z_2^2}$$

$$T_2 = -\frac{9z_2\left(T_0^1 + z_1T_1^1 + z_2T_2^1\right)}{4z_1^3 + 27z_2^2} + \frac{6z_1\left(T_0^2 + z_1T_1^2 + z_2T_2^2\right)}{4z_1^3 + 27z_2^2}$$

$$\widehat{\mathbf{b}}_1 = -\frac{2z_1^2\left(\widehat{\mathbf{C}}_0^1 + z_1\widehat{\mathbf{C}}_1^1 + z_2\widehat{\mathbf{C}}_2^1\right)}{4z_1^3 + 27z_2^2} - \frac{9z_2\left(\widehat{\mathbf{C}}_0^2 + z_1\widehat{\mathbf{C}}_1^2 + z_2\widehat{\mathbf{C}}_2^2\right)}{4z_1^3 + 27z_2^2}$$

$$\widehat{\mathbf{b}}_2 = -\frac{9z_2\left(\widehat{\mathbf{C}}_0^1 + z_1\widehat{\mathbf{C}}_1^1 + z_2\widehat{\mathbf{C}}_2^1\right)}{4z_1^3 + 27z_2^2} + \frac{6z_1\left(\widehat{\mathbf{C}}_0^2 + z_1\widehat{\mathbf{C}}_1^2 + z_2\widehat{\mathbf{C}}_2^2\right)}{4z_1^3 + 27z_2^2}.$$

The compatibility equation (13) gives

$$T_0^2 = T_1^2 = T_2^1 = 2T_2^2 - 3T_1^1 = 0 \tag{25}$$

and equations (14) are

$$\begin{aligned}
&\left[\widehat{\mathbf{C}}_2^1, \widehat{\mathbf{C}}_2^2\right] = 0,\\
&\left[\widehat{\mathbf{C}}_1^1, \widehat{\mathbf{C}}_2^2\right] + \left[\widehat{\mathbf{C}}_2^1, \widehat{\mathbf{C}}_1^2\right] = 0,\\
&\left[\widehat{\mathbf{C}}_1^1, \widehat{\mathbf{C}}_1^2\right] = \tfrac{2}{3}\widehat{\mathbf{C}}_2^1,\\
&\left[\widehat{\mathbf{C}}_0^1, \widehat{\mathbf{C}}_2^2\right] + \left[\widehat{\mathbf{C}}_2^1, \widehat{\mathbf{C}}_0^2\right] = 2\widehat{\mathbf{C}}_2^2 - 3\widehat{\mathbf{C}}_1^1,\\
&\left[\widehat{\mathbf{C}}_0^1, \widehat{\mathbf{C}}_1^2\right] + \left[\widehat{\mathbf{C}}_1^1, \widehat{\mathbf{C}}_0^2\right] = \widehat{\mathbf{C}}_1^2,\\
&\left[\widehat{\mathbf{C}}_0^1, \widehat{\mathbf{C}}_0^2\right] = -\widehat{\mathbf{C}}_0^2.
\end{aligned} \tag{26}$$

If we write conditions (25) in the form $T_0^1 = \gamma$, $T_1^1 = 2\omega$ and $T_2^2 = 3\omega$, we obtain

$$T_1 = \frac{-2\gamma z_1^2}{4z_1^3 + 27z_2^2} - \omega, \qquad T_2 = \frac{-9\gamma z_2}{4z_1^3 + 27z_2^2}$$

and equations (15) are

$$\partial_1 F = \frac{-2(\gamma + 1)z_1^2}{4z_1^3 + 27z_2^2} - \omega, \qquad \partial_2 F = \frac{-9(\gamma + 1)z_2}{4z_1^3 + 27z_2^2},$$

from which it follows that

$$F(z_1, z_2) = -\frac{\gamma+1}{6}\ln\left|4z_1^3 + 27z_2^2\right| - \omega z_1.$$

Thus, we have

$$\widehat{U}^{(1)} = g^{rs}(F_{,rs} + F_{,r}F_{,s} - \Gamma_s F_{,r}) = -2\gamma\omega - 2\omega^2 z_1 - \frac{2(\gamma+1)(\gamma+4)z_1^2}{4z_1^3 + 27z_2^2}$$

and part of the relative potential in the case $\mathbf{b}^r = 0$ is

$$U_{\mathrm{rel}}^{(1)} = C + \omega^2\left(y_1^2 + y_1y_2 + y_2^2\right) + \frac{(\gamma+1)(\gamma+4)}{9}\left(\frac{1}{(y_1-y_2)^2} + \frac{1}{(2y_1+y_2)^2} + \frac{1}{(y_1+2y_2)^2}\right), \tag{27}$$

where C, γ and ω are any constants. In the case of the scalar model, i.e. $N = 1$, we obtain by this method the known Calogero model [3].

For a general matrix potential we have to find a solution of the system (26). One of the solutions of this system for general N is given by

$$\widehat{\mathbf{C}}_1^1 = \widehat{\mathbf{C}}_2^1 = \widehat{\mathbf{C}}_2^2 = 0 \qquad \left[\widehat{\mathbf{C}}_0^1, \widehat{\mathbf{C}}_1^2\right] = \widehat{\mathbf{C}}_1^2 \qquad \text{and} \qquad \left[\widehat{\mathbf{C}}_0^1, \widehat{\mathbf{C}}_0^2\right] = -\widehat{\mathbf{C}}_0^2. \tag{28}$$

In this case the functions $\widehat{\mathbf{b}}^r$ are

$$\widehat{\mathbf{b}}^1 = \widehat{\mathbf{C}}_0^1 \qquad \text{and} \qquad \widehat{\mathbf{b}}^2 = \widehat{\mathbf{C}}_0^2 + \widehat{\mathbf{C}}_1^2 z_1$$

and the part of the potential $\widehat{\mathbf{U}}$ which involves the matrices $\widehat{\mathbf{b}}^r$ is

$$\begin{aligned}
\widehat{\mathbf{U}}^{(m)} &= \partial_r\widehat{\mathbf{b}}^r + g_{rs}\widehat{\mathbf{b}}^r\widehat{\mathbf{b}}^s + \Gamma_{rs}^s\widehat{\mathbf{b}}^r + 2\widehat{\mathbf{b}}^r F_{,r} - \mathbf{V} \\
&= -\widehat{\mathbf{V}} - 2\omega\widehat{\mathbf{C}}_0^1 + \frac{1}{4z_1^3 + 27z_2^2} \times \Big[6z_1^3\big(\widehat{\mathbf{C}}_1^2\big)^2 + 6z_1\big(\widehat{\mathbf{C}}_0^2\big)^2 \\
&\quad - 2z_1^2\Big(\big(\widehat{\mathbf{C}}_0^1\big)^2 + (2\gamma+5)\widehat{\mathbf{C}}_0^1 - 3\big(\widehat{\mathbf{C}}_0^2\widehat{\mathbf{C}}_1^2 + \widehat{\mathbf{C}}_1^2\widehat{\mathbf{C}}_0^2\big)\Big) \\
&\quad - 9z_1z_2\big(\widehat{\mathbf{C}}_0^1\widehat{\mathbf{C}}_1^2 + \widehat{\mathbf{C}}_1^2\widehat{\mathbf{C}}_0^1 + (2\gamma+5)\widehat{\mathbf{C}}_1^2\big) \\
&\quad - 9z_2\big(\widehat{\mathbf{C}}_0^1\widehat{\mathbf{C}}_0^2 + \widehat{\mathbf{C}}_0^2\widehat{\mathbf{C}}_0^1 + (2\gamma+5)\widehat{\mathbf{C}}_0^2\big)\Big].
\end{aligned} \tag{29}$$

The system of partial differential equations (16) for the matrices $\widehat{\mathbf{G}}$ now has the form

$$\begin{aligned}
\frac{\partial\widehat{\mathbf{G}}}{\partial z_1} &= -\frac{2z_1^2}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^1 - \frac{9z_2}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^2 - \frac{9z_1z_2}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_1^2, \\
\frac{\partial\widehat{\mathbf{G}}}{\partial z_2} &= -\frac{9z_2}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^1 + \frac{6z_1}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^2 + \frac{6z_1^2}{4z_1^3 + 27z_2^2}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_1^2,
\end{aligned} \tag{30}$$

or on the variables y_1, y_2

$$\begin{aligned}
\frac{\partial\widehat{\mathbf{G}}}{\partial y_1} &= \frac{y_2^2 - 2y_1y_2 - 2y_1^2}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^1 + \frac{3y_2}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^2 - \frac{3y_2\left(y_1^2 + y_1y_2 + y_2^2\right)}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_1^2 \\
\frac{\partial\widehat{\mathbf{G}}}{\partial y_2} &= \frac{2y_2^2 + 2y_1y_2 - y_1^2}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^1 - \frac{3y_1}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_0^2 + \frac{3y_1\left(y_1^2 + y_1y_2 + y_2^2\right)}{D}\widehat{\mathbf{G}}\widehat{\mathbf{C}}_1^2,
\end{aligned} \tag{31}$$

where $D = (y_1 - y_2)(2y_1 + y_2)(y_1 + 2y_2)$.

3.1. Models with matrices of the type 2×2

For the matrix of the type 2×2 we can choose the solution of (28) as

$$\widehat{\mathbf{C}}_0^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\mathbf{e}_0, \qquad \widehat{\mathbf{C}}_1^2 = \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} = \rho\mathbf{e}_{12}, \qquad \widehat{\mathbf{C}}_0^2 = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix} = \sigma\mathbf{e}_{21},$$

where ρ and σ are real constants. The transformed matrix potential (29) in this case is

$$\widehat{\mathbf{U}}^{(2)} = -\omega\mathbf{e}_0 - \frac{(1-12\rho\sigma)z_1^2}{2\left(4z_1^3 + 27z_2^2\right)} - \frac{2\gamma+5}{4z_1^3+27z_2^2}\left(z_1^2\mathbf{e}_0 + 9\sigma z_2\mathbf{e}_{21} + 9\rho z_1 z_2\mathbf{e}_{12}\right) - \mathbf{V}. \tag{32}$$

To obtain a nontrivial case, the conditions for the symmetry of the potential $\mathbf{U}_{\mathrm{rel}}$ give

$$\mathbf{V} + \omega\mathbf{e}_0 = 0 \qquad \text{and} \qquad \rho\sigma = -\tfrac{2}{3}$$

and the corresponding transformed potential (32) is

$$\widehat{\mathbf{U}}^{(2)} = -\frac{9z_1^2}{2\left(4z_1^3+27z_2^2\right)} - \frac{2\gamma+5}{4z_1^3+27z_2^2}\begin{pmatrix} z_1^2 & 9\rho z_1 z_2 \\ -\frac{6}{\rho}z_2 & -z_1^2 \end{pmatrix}.$$

When we write $\widehat{\mathbf{G}} = \left(\begin{smallmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{smallmatrix}\right)$, the system of the partial differential equations (31) for $\widehat{\mathbf{G}}$ is

$$\frac{\partial G_{k1}}{\partial y_1} = \frac{\left(y_2^2 - 2y_1y_2 - 2y_1^2\right)G_{k1}}{2(y_1-y_2)(2y_1+y_2)(y_1+2y_2)} - \frac{2y_2G_{k2}}{\rho(y_1-y_2)(2y_1+y_2)(y_1+2y_2)}$$

$$\frac{\partial G_{k1}}{\partial y_2} = \frac{\left(2y_2^2 + 2y_1y_2 - y_1^2\right)G_{k1}}{2(y_1-y_2)(2y_1+y_2)(y_1+2y_2)} + \frac{2y_1G_{k2}}{\rho(y_1-y_2)(2y_1+y_2)(y_1+2y_2)}$$

$$\frac{\partial G_{k2}}{\partial y_1} = -\frac{3\rho y_2\left(y_1^2 + y_1y_2 + y_2^2\right)G_{k1}}{(y_1-y_2)(2y_1+y_2)(y_1+2y_2)} + \frac{\left(2y_1^2 + 2y_1y_2 - y_2^2\right)G_{k2}}{2(y_1-y_2)(2y_1+y_2)(y_1+2y_2)}$$

$$\frac{\partial G_{k2}}{\partial y_2} = \frac{3\rho y_1\left(y_1^2 + y_1y_2 + y_2^2\right)G_{k1}}{(y_1-y_2)(2y_1+y_2)(y_1+2y_2)} + \frac{\left(y_1^2 - 2y_1y_2 - 2y_2^2\right)G_{k2}}{2(y_1-y_2)(2y_1+y_2)(y_1+2y_2)},$$

where $k = 1, 2$. The general regular solution of this system is

$$\widehat{\mathbf{G}} = D^{-1/2}\mathbf{C}\begin{pmatrix} 2(2y_1+y_2) & 3\rho y_2(2y_1+y_2) \\ 2(y_1+2y_2) & 3\rho y_1(y_1+2y_2) \end{pmatrix},$$

where $\mathbf{C}$ is a constant regular matrix. This matrix has to be chosen so that the matrix

$$\widehat{\mathbf{G}}\begin{pmatrix} \left(y_1^2+y_1y_2+y_2^2\right)^2 & 9\rho y_1y_2\left(y_1+y_2\right)\left(y_1^2+y_1y_1+y_2^2\right) \\ \frac{6}{\rho}y_1y_2\left(y_1+y_2\right) & -\left(y_1^2+y_1y_2+y_2^2\right)^2 \end{pmatrix}\widehat{\mathbf{G}}^{-1}$$

$$= \mathbf{C}\begin{pmatrix} (y_1+y_2)(y_1-y_2)^3 & y_2(2y_1+y_2)^3 \\ y_1(y_1+2y_2)^3 & -(y_1+y_2)(y_1-y_2)^3 \end{pmatrix}\mathbf{C}^{-1}$$

can be symmetric.

The solution of these equations is[3]

$$\mathbf{C} = \begin{pmatrix} 3^{-1/4} & 3^{-1/4} \\ -3^{1/4} & 3^{1/4} \end{pmatrix}.$$

If we use this matrix $\mathbf{C}$, we obtain the symmetric potential

$$\mathbf{U}^{(2)} = \frac{1}{2}\left(\frac{1}{(y_1-y_2)^2} + \frac{1}{(y_1+2y_2)^2} + \frac{1}{(2y_1+y_2)^2}\right) + \frac{2\gamma+5}{3}\begin{pmatrix} U & V \\ V & -U \end{pmatrix},$$

[3] All other solutions of these equations have the form $\lambda\mathbf{RC}$, where λ is a real constant, $\mathbf{R}$ is any orthogonal matrix, and $\mathbf{C}$ the matrix given in the main text.

where

$$\begin{aligned} U &= \frac{1}{2}\left(\frac{2}{(y_1-y_2)^2} - \frac{1}{(y_1+2y_2)^2} - \frac{1}{(2y_1+y_2)^2}\right) \\ V &= \frac{\sqrt{3}}{2}\left(\frac{1}{(2y_1+y_2)^2} - \frac{1}{(y_1+2y_2)^2}\right). \end{aligned} \tag{33}$$

And finally, the potential of the model with the 2×2 matrix potential is

$$\begin{aligned} \mathbf{U}_{\mathrm{rel}} = U^{(1)}_{\mathrm{rel}} + \frac{1}{2}\mathbf{U}^{(2)} &= C + \omega^2\big(y_1^2 + y_1y_2 + y_2^2\big) + \frac{(2\gamma+5)^2}{36} \\ &\times \left(\frac{1}{(y_1-y_2)^2} + \frac{1}{(y_1+2y_2)^2} + \frac{1}{(2y_1+y_2)^2}\right) + \frac{2\gamma+5}{6}\begin{pmatrix} U & V \\ V & -U \end{pmatrix}. \end{aligned}$$

The model with this matrix potential was found by a different method in [4].

3.2. *Models with matrices of the type* 3×3

The solution of equations (28) is

$$\widehat{\mathbf{C}}^1_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \widehat{\mathbf{C}}^2_0 = \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}, \qquad \widehat{\mathbf{C}}^2_1 = \begin{pmatrix} 0 & \rho_1 & 0 \\ 0 & 0 & \rho_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

We suppose the relations $\rho_1\sigma_1 = \rho_2\sigma_2 = -\frac{4}{3}$. This assumption simplifies equations (31). In this case, equation (29) has the form

$$\widehat{\mathbf{U}}^{(3)} = -\mathbf{V} - 2\omega\widehat{\mathbf{C}}^1_0 - \frac{12z_1^2}{4z_1^3 + 27z_2^2} - \frac{2}{4z_1^3 + 27z_2^2}\begin{pmatrix} \widehat{U}_{11} & \widehat{U}_{12} & \widehat{U}_{13} \\ \widehat{U}_{21} & \widehat{U}_{22} & \widehat{U}_{23} \\ \widehat{U}_{31} & \widehat{U}_{32} & \widehat{U}_{33} \end{pmatrix},$$

where

$$\begin{aligned} &\widehat{U}_{11} = 2(\gamma+2)z_1^2, && \widehat{U}_{12} = 9(\gamma+3)\rho_1 z_1 z_2, && \widehat{U}_{13} = -3\rho_1\rho_2 z_1^3, \\ &\widehat{U}_{21} = -\frac{12(\gamma+3)}{\rho_1}z_2, && \widehat{U}_{22} = 2z_1^2, && \widehat{U}_{23} = 9(\gamma+2)\rho_2 z_1 z_2, \\ &\widehat{U}_{31} = -\frac{16}{3\rho_1\rho_2}z_1, && \widehat{U}_{32} = -\frac{12(\gamma+2)}{\rho_2}z_2, && \widehat{U}_{33} = -2(\gamma+3)z_1^2. \end{aligned}$$

To obtain the model, which can be symmetrized, we have to take $\mathbf{V} = -2\omega\widehat{\mathbf{C}}^1_0 + \frac{3}{2}\rho_1\rho_2\mathbf{e}_{13}$. One solution of (31) which give the symmetric potential is

$$\widehat{\mathbf{G}} = \frac{1}{D}\begin{pmatrix} \widehat{G}_{11} & \widehat{G}_{12} & \widehat{G}_{13} \\ \widehat{G}_{21} & \widehat{G}_{22} & \widehat{G}_{23} \\ \widehat{G}_{31} & \widehat{G}_{32} & \widehat{G}_{33} \end{pmatrix},$$

where

$$D = (y_1 - y_2)(2y_1 + y_2)(2y_2 + y_1)$$

and

$$\widehat{G}_{11} = \frac{12}{\sqrt{2\gamma+5}}\big(3(y_1+y_2)^2\gamma + 7y_1^2 + 16y_1y_2 + 7y_2^2\big)$$

$$\widehat{G}_{12} = \frac{18\rho_1}{\sqrt{2\gamma+5}}(y_1+y_2)\big(\big(y_1^2 + 4y_1y_2 + y_2^2\big)\gamma + 3\big(y_1^2 + 3y_1y_2 + y_2^2\big)\big)$$

$$\widehat{G}_{13} = \frac{9\rho_1\rho_2}{2\sqrt{2\gamma+5}}\left(\left(y_1^2+4y_1y_2+y_2^2\right)^2\gamma + y_1^4 + 20y_1^3y_2 + 48y_1^2y_2^2 + 20y_1y_2^3 + y_2^4\right)$$
$$\widehat{G}_{21} = -12\sqrt{3(\gamma+3)}(y_1-y_2)(y_1+y_2)$$
$$\widehat{G}_{22} = 6\rho_1\sqrt{3(\gamma+3)}(y_1-y_2)\left(y_1^2+y_1y_2+y_2^2\right)$$
$$\widehat{G}_{23} = \frac{9}{2}\rho_1\rho_2\sqrt{3(\gamma+3)}(y_1-y_2)(y_1+y_2)\left(y_1^2+4y_1y_2+y_2^2\right)$$
$$\widehat{G}_{31} = 12\sqrt{\frac{(\gamma+2)(\gamma+3)}{2\gamma+5}}(y_1-y_2)^2$$
$$\widehat{G}_{32} = -18\rho_1\sqrt{\frac{(\gamma+2)(\gamma+3)}{2\gamma+5}}(y_1+y_2)(y_1-y_2)^2$$
$$\widehat{G}_{33} = \frac{27}{2}\rho_1\rho_2\sqrt{\frac{(\gamma+2)(\gamma+3)}{2\gamma+5}}(y_1+y_2)^2(y_1-y_2)^2.$$

With the use of this matrix we obtain the matrix part of the potential

$$\begin{aligned}\mathbf{U}^{(3)} &= \widehat{\mathbf{G}}\widehat{\mathbf{U}}^{(3)}\widehat{\mathbf{G}}^{-1}\\ &= \frac{4}{3}\left(\frac{1}{(y_1-y_2)^2}+\frac{1}{(2y_1+y_2)^2}+\frac{1}{(y_1+2y_2)^2}\right)+\frac{4}{3}\begin{pmatrix}U_{11} & U_{12} & 0\\ U_{12} & U_{22} & U_{23}\\ 0 & U_{23} & U_{33}\end{pmatrix},\end{aligned}$$

where

$$U_{11} = (\gamma+3)U, \qquad U_{22} = -U, \qquad U_{33} = -(\gamma+2)U$$
$$U_{12} = (\gamma+1)\sqrt{\frac{\gamma+3}{2\gamma+5}}V, \qquad U_{23} = (\gamma+4)\sqrt{\frac{\gamma+2}{2\gamma+5}}V$$

and U, V are given by relations (33). Finally, the relative potential of this matrix model is

$$\begin{aligned}\mathbf{U}_{\mathrm{rel}} = U_{\mathrm{rel}}^{(1)} + \frac{1}{2}\mathbf{U}^{(3)} &= C + \omega^2\left(y_1^2+y_1y_2+y_2^2\right) + \frac{\gamma^2+5\gamma+10}{9}\\ &\quad\times\left(\frac{1}{(y_1-y_2)^2}+\frac{1}{(2y_1+y_2)^2}+\frac{1}{(y_1+2y_2)^2}\right)+\frac{2}{3}\begin{pmatrix}U_{11} & U_{12} & 0\\ U_{12} & U_{22} & U_{23}\\ 0 & U_{23} & U_{33}\end{pmatrix}.\end{aligned}$$

4. Conclusions

In the paper, the method of construction of matrix solvable models is presented. The method is illustrated on the A_2 system of the Calogero type for three particles. For a comprehensive review of these systems connected with different root systems see [5].

At this moment there exist many [6–11] matrix models. The method of construction is connected with Dunkel operators [12] and using in the A_n case the representations of algebra $\mathcal{A}$

$$\mathbf{M}_{ij}^2 = 1, \qquad \mathbf{M}_{ij}\mathbf{M}_{jl} = \mathbf{M}_{il}\mathbf{M}_{ij} = \mathbf{M}_{jl}\mathbf{M}_{il}, \qquad [\mathbf{M}_{ij}, \mathbf{M}_{lm}] = 0,$$

where i, j, l, m are different indices. In our case, A_2 $i, j, l, m = 1, 2, 3$.

In the 2×2-matrix case, it is easy to show that the two-body Calogero interaction

$$\gamma(\gamma \pm \mathbf{M}_{ij})$$

gives our model, if we take representation of $\mathcal{A}$ by operators

$$M_{12} = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad M_{13} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_{23} = -\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}.$$

A simple analysis shows that three-dimensional irreducible representations of algebra $\mathcal{A}$ do not exist. In our case, the potential of the model is not reducible. Thus, our model is new.

The aim of the above examples was to demonstrate the method. In the following paper we prepare the explicit calculation of the energy eigenvalues and eigenfunctions for our new model starting from the solvability. For this calculation knowledge of the integral of motion is not needed.

It is evident that the method is applicable to many general examples. The first substantial generalization is to obtain the models for three particles with $N \times N$ matrices. The results have already been obtained, and all these models are new in the sense mentioned above.

The second generalization respects the number of particles n, which is a really physically interesting problem. The first step to solution of this problem was published in paper [13], where the solvability of the n particle Calogero model and connection with the realization of the algebra $\mathrm{gl}(n)$ was shown in the non-matrix case. This will fix the matrix tensor g^{rs} and a change of variables. The explicit analysis of equation (14) can be performed, so the only open problem is to find an effective method for the solution of equations (16). Anyway, as we can see from our 3×3 matrix example for three particles, many of the models obtained will be new, and there is good reason to make such cumbersome calculations for the n particle systems as well.

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References

[1] Turbiner A V 1998 *Mod. Phys. Lett.* A **13** 1473–83
[2] Shifman M A and Turbiner A V 1989 *Commun. Math. Phys.* **126** 347–65
[3] Calogero F 1969 *J. Math. Phys.* **10** 2191, 219
Calogero F 1971 *J. Math. Phys.* **12** 419
[4] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703–5 (*Preprint* hep-th/9206046)
[5] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 314
Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 6
[6] Minahan J A and Polychronakos A P 1993 *Phys. Lett.* B **302** 265
[7] Yamamoto T 1995 *Phys. Lett.* A **208** 293
[8] Basu-Mallick B 1996 *Nucl. Phys.* B **482** 713
[9] Finkel F, Gómez-Ullate D, González-López A, Rodríguez M A and Zhdanov R 2001 *Commun. Math. Phys.* **221** 477
[10] Finkel F, Gómez-Ullate G, González-López A, Rodríguez M A and Zhdanov R 2001 *Nucl. Phys.* B **613** 472
[11] Enciso A, Finkel F, González-López A and Rodríguez M A 2004 *Preprint* hep-th/0406054
[12] Dunkl F C 1989 *Trans. Am. Math. Soc.* **311** 167
[13] Rühl W and Turbiner A 1995 *Mod. Phys. Lett.* A **10** 2213–22